12. Präsenzübung, **Statistische Physik**

zu bearbeiten am Donnerstag, 12.1.2012

Aufgabe P26 *Kramers-Wannier duality*

We consider the translation-invariant Ising model on a two-dimensional square lattice with periodic boundary conditions and zero external field. The Hamiltonian is

$$H = -J \sum_{ij} \sigma_i \sigma_j$$

where $J > 0$, $\sigma_i \in \{-1, +1\}$ and the sum is over all pairs of nearest neighbors (called *edges*). For instance, any given spin σ_i appears in four terms of the sum, one for each of its four nearest neighbors. Let N be the total number of spins, and $L = 2N$ the total number of edges. We know that this system has a phase transition at a critical temperature τ_c. At zero temperature the spins all have the same value, say $\sigma_i = +1$. This configuration is called one (of two) ground state(s).

a. Write down the energy of the ground states in terms of N and J.

b. Consider the ordered phase at $\tau < \tau_c$. Any configuration can be reached by taking one of the ground states and flipping a subset of all spins. Note that some of those spins may be contiguous. The energy E_l of this configuration only depends of the number l of edges bounding the regions of flipped spins (you may compute E_l for some examples to understand this). Let $g(l)$ be the number of possible subsets of spins with a given number l of bounding edges. For instance, $g(0) = 2$ and the next nonzero value is $g(4) = 2N$ (corresponding to flipping either 1 spin or $N - 1$ spins). Write down the partition function Z as a sum over the integer l. Why can this be considered as a low temperature expansion of Z?

c. We now want a high temperature expansion of Z. Show that, given that the spins only take values $+1$ or -1, then

$$e^{J\sigma_i\sigma_j/\tau} = \cosh(J/\tau)(1 + w\sigma_i\sigma_j)$$

where $w := \tanh(J/\tau)$. Note that w is small when τ is large. Using the above result, write down the partition function as a power series in w.

Hint: it involves the same function $g(l)$ used in point b.

d. In the thermodynamic limit ($N \to \infty$), the two different expressions obtained for Z, although formally exact, may not be valid for all temperatures because the series may not converge. However, we expect the first one to be valid for small temperatures, and the second one to be valid for high temperature.

In fact, for any N, the series involved in the low-temperature expansion of Z at temperature τ is equal to that involved in the high-temperature expansion of Z at a dual temperature τ^*. Find the connection between τ and τ^*, and between the corresponding partition functions $Z(\tau)$ and $Z(\tau^*)$. Also show that $\tau^* \to \infty$ as $\tau \to 0$.

e. Show that the temperature τ_0 such that $\tau_0 = \tau_0^*$ is

$$\tau_0 = 2J/\ln(1 + \sqrt{2}).$$

This turns out to be the critical temperature: $\tau_c = \tau_0$.